Bargmann-Michel-Telegdi equation and one-particle relativistic approach

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Abstract

A reexamination of the semiclassical approach of the relativistic electron indicates a possible variation of its helicity for electric and magnetic static fields applied along its global motion due to zitterbewegung effects, proportional to the anomalous part of the magnetic moment.

1 Introduction

The one-particle interpretation of Dirac equation is an old problem which has not a unique solution. The usual Foldy-Wouthuysen method [1] is suitable for atomic physics where the electron velocity is small but cannot be applied in presence of intense electromagnetic fields. On the other hand the semiclassical limit [2] of Dirac equation when the particle wavelength is small compared to the characteristic distance of the electromagnetic potential leads to a trajectory determined by the Lorentz force plus a spin precession given by the Bargmann-Michel-Telegdi equation [3] (BMT). The latter approximation is better for relativistic velocities [4] and might be applied also for particles in laser fields [5].

The purpose of the present work is to allow the influence in the semiclassical approximation of the motion of the electron inside the scale of the Compton length and the perturbation caused by the zero-point radiation field.

Whereas non-relativistic quantum mechanics has equations of motion for operators equivalent to the classical Hamilton ones, Dirac equation leads to the picture that mass and charge do not coincide. This has suggested classical models [6] which double the number of variables and correspond to equations of motion equivalent to the operatorial Dirac ones, giving the spin as a result of zitterbewegung. This analogy indicates that zitterbewegung is twofold: that of mass and a broader one of charge. On the other hand this classical model does not give the average of velocities which emerges from the semiclassical approximation of the operatorial Dirac equations. As a consequence we refer to BMT which involves the average of only one velocity, taking it as the effective common charge-mass one.

We have therefore reinterpreted BMT in such a way that, keeping its feature of affecting "Thomas $\frac{1}{2}$" in the spin orbit interaction only to Dirac magnetic moment but not to its anomalous correction [7], allows a semi-classical explanation of the latter. This is so because, instead of the view that the fluctuations of the zero-point radiation field can only decrease the effective spin [8], the velocity contribution to spin in BMT increases by this fluctuation added to the zitterbewegung one [9].

The main result of the present paper is that the zero component of the Pauli-Lubansky polarization vector defined as average of the instantaneous variables may not vanish in the electron global rest frame because of the zitterbewegung. This is suggested both by the semiclassical limit [10] of the Dirac equation and its classical analogy [6].

A consequence of this would be the appearance of an additional term to the spin-orbit interaction, extremely tiny even for very strong magnetic fields applied to atoms, proportional to the anomalous pact of magnetic moment. But this consequence may be doubtful because of the low velocity of the electron in an atom. Another effect, also proportional to the radiative correction to magnetic moment. which might be more trustful corresponds to a small variation with time of the helicity of a free electron to which strong longitudinal electric and magnetic fields are applied. This effect resembles the semiclassical interpretation of chiral anomaly using its equivalence to the explicit symmetry breaking for a very massive particle [11] in a model in which the particle spin is entirely due to precession around a magnetic field [12]. The relation of the anomaly effect on the electron propagation with its anomalous moment correction has also been noticed at the level of Feynman diagrams [13]. It must be remarked that our effect due to longitudinal fields is different from the transverse polarization by radiation of electrons in storage rings [14].

2 Reinterpretation of BMT

We remind that normal BMT is

$$\frac{da^\mu}{d\tau} = c_1 F^{\mu\nu} a_\nu + c_2 \frac{\pi^\mu}{m}\frac{\pi_\nu}{m} F^{\nu\lambda} a_\lambda \tag{1}$$

where π^ν is the kinetic momentum and a^μ the Pauli-Lubansky vector

$$a^\mu = \epsilon^{\mu\nu\rho\sigma} \frac{\pi_\nu}{m} M_{\rho\sigma} \qquad . \tag{2}$$

By definition $\pi \cdot a = 0$ and in the rest frame a^μ reduces to a 3-vector $\boldsymbol{\xi}$ which is interpreted as twice the average of spin. τ is the proper time and m the rest mass.

From eq.1 in rest frame for spatial indices $c_1 = 2\mu$, where μ is the magnetic moment. Constancy of $\pi \cdot a = 0$ in any frame together with Lorentz force

$$\frac{d\pi^\nu}{d\tau} = eF^{\mu\nu} \frac{\pi_\nu}{m} \tag{3}$$

leads to $-c_2 = 2\mu' = 2(\mu - \frac{e}{2m})$. In rest frame $\frac{d}{d\tau}(\boldsymbol{\xi}^2) = 0$ and, in general, $\frac{d}{d\tau}(a^2) = 0$.

Our interpretation of BMT is based on considering it by covariance

$$\frac{da^\mu}{d\tau} = \tilde{c}_1 F^{\mu\nu} a_\nu + \tilde{c}_2 \langle u^\mu u_\nu \rangle F^{\nu\lambda} a_\lambda + \tilde{c}_3 \frac{\pi^\mu}{m}\frac{\pi_\nu}{m} F^{\nu\lambda} a_\lambda \tag{4}$$

where the second term contains the quadratic average of common charge and mass instantaneous velocity u^μ different from the global π^μ/m.

In rest frame for only magnetic field just the first two terms of eq.4 contribute to the evolution of $\boldsymbol{\xi}$. Since we assume that, in absence of radiative corrections, the second term represents the common charge-mass zitterbewegung it must give one Bohr magneton corresponding to the ratio of magnetic moment to angular momentum due to proportional density of charge and mass. Consequently, the first term represents the effect of zitterbewegung of charge relative to mass and must account for the other Bohr magneton of Dirac spin. Therefore

$$\tilde{c}_1 = \frac{e}{2m} \qquad , \qquad -\tilde{c}_2 \frac{1}{3} \langle \mathbf{u}^2 \rangle_Z = \frac{e}{2m} \qquad . \tag{5}$$

The additional radiative correction gives the right first-order anomalous magnetic moment if we choose the zitterbewegung average

$$\langle \mathbf{u}^2 \rangle_Z = 1 = \langle \frac{\mathbf{v}^2}{1-\mathbf{v}^2} \rangle \qquad . \tag{6}$$

In fact, in rest frame, the ordinary semiclassical treatment of zero-point radiation effect [15] for wavelengths larger than the Compton one produces, in addition to the above average of $\mathbf{u}^2$ another

$$\langle \mathbf{u}^2 \rangle_R = \frac{\alpha}{\pi} \tag{7}$$

considering that only quadratic variables are affected by fluctuations. To obtain eq.7 only the UV cut off $\omega_c = m$ must be introduced without IR cutoff in agreement with what occurs in QED. If we express in covariant way the quadratic average of velocities such that the above rest result is reproduced, we must write

$$\langle u^\mu u^\nu \rangle = -\frac{1}{3}(1+\frac{\alpha}{\pi})g^{\mu\nu} + \frac{\pi^\mu}{m}\frac{\pi^\nu}{m} \qquad . \tag{8}$$

In order that eq.4 be equal to BMT eq.1, which is necessary to have Thomas $\frac{1}{2}$ only in normal magnetic moment and not in its anomalous part in spin orbit interaction, we choose $c_3 = 3\frac{e}{2m} - 2\mu'$ so that

$$\frac{da^\mu}{d\tau} = 2\frac{e}{2m}(1+\frac{\alpha}{2\pi})F^{\mu\nu}a_\nu - 2\mu'\frac{\pi^\mu}{m}\frac{\pi_\nu}{m}F^{\nu\lambda}a_\lambda \qquad . \tag{9}$$

The difference is now that since we consider instantaneous u^μ and global $\frac{\pi^\mu}{m}$ velocities there is an arbitrariness in the definition of Pauli-Lubansky vector according to which one we introduce in eq.2 and it is not obvious that $\pi \cdot a = 0$. If this orthogonality does not hold, in the rest frame there is a ξ_0 apart from $\boldsymbol{\xi}$ to characterize the polarization.

It is interesting that something similar regarding ξ_0 emerges from the semiclassical limit of Dirac equation [10]. In fact, taking $\hbar \to 0$ for Dirac equation for Ψ in presence of electromagnetic field with the definition

$$\Psi = \Phi R \;\; exp\frac{i}{\hbar}S \tag{10}$$

where R and S are real functions and Φ is a complex 4-spinor with the conditions

$$\bar{\Phi}\Phi = 1 \qquad , \qquad \bar{\Phi}\gamma^5\Phi = 0 \tag{11}$$

the equation of order $\hbar$ for Φ shows that with the identification

$$\frac{\pi^\mu}{m} = \bar{\Phi}\gamma^\mu\Phi \qquad , \qquad a^\mu = \bar{\Phi}\gamma^\mu\gamma_5\Phi \tag{12}$$

π^μ and a^μ satisfy the Lorentz force and BMT with $\mu' = 0$ respectively.

Now in the global rest frame

$$\xi_0 = \Phi^\dagger\gamma_5\Phi \tag{13}$$

is in general non-vanishing in presence of electromagnetic fields because Dirac equation couples $p_\mu - eA_\mu$ to the zitterbewegung of charge and a non-vanishing weak bispinor of Φ may survive. We must note that, since

$$\gamma_\mu\gamma_5 = -\frac{1}{3!}\epsilon_{\mu\nu\rho\tau}\gamma^\nu\sigma^{\rho\tau} \qquad ,$$

this possibility of non-vanishing ξ_0 is a consequence of having defined the Pauli-Lubansky vector with instantaneous velocity and not with the global one.

An analogy of the Dirac zitterbewegung is given by the classical Lagrangian [6]

$$L = -i\bar{z}\dot{z} + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) + eA_\mu\bar{z}\gamma^\mu z \tag{14}$$

where z is a 4-spinor.

Defining

$$v^\mu \equiv \dot{x}^\mu = \bar{z}\gamma^\mu z \qquad , \qquad S^{\mu\nu} = -\frac{i}{4}\bar{z}[\gamma^\mu, \gamma^\nu]z \tag{15}$$

together with $P_\mu = p_\mu - eA_\mu$, the classical equations of motion

$$\begin{aligned} \frac{dP_\mu}{d\tau} &= eF_{\mu\nu}v^\nu \\ \frac{dv_\mu}{d\tau} &= -4S_{\mu\nu}P^\nu \\ \frac{dS_{\mu\nu}}{d\tau} &= v_\mu P_\nu - P_\mu v_\nu \end{aligned} \tag{16}$$

are equal to those obtained for the operators $p_\mu - eA_\mu$, γ^μ and $[\gamma^\mu, \gamma^\nu]$ from the Dirac Hamiltonian with the definition

$$\frac{d}{d\tau}O = -i\gamma^0[O, H_D] \qquad . \tag{17}$$

Whereas the contraction

$$v \cdot P = m \tag{18}$$

valid for the classical analogy is also a constant as a Dirac operator, the modulus of the mass velocity $\frac{P^\mu}{m}$ and charge velocity v^ν are not constants

$$\frac{d}{d\tau}v^2 \neq 0 \qquad , \qquad \frac{d}{d\tau}P^2 \neq 0 \tag{19}$$

indicating that both are subject of zitterbewegung.

In the quantum case obviously $\gamma^\mu\gamma_\mu$ is a constant. The difference being that 4x4 matrices take into account also negative energy contributions whereas eq.19 refers to a positive energy particle.

If in the classical analogy we wish to define a spin 4-vector as in eq.2 we have the arbitrariness of using either the mass or the charge velocity. Choosing the former for the definition of the spin 4-vector and the latter for its inverse, i.e.

$$\begin{aligned} S^\mu &= \epsilon^{\mu\nu\rho\sigma}\frac{P_\nu}{m}S_{\rho\sigma} \\ S^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma \end{aligned} \tag{20}$$

which are compatible, we obtain

$$\frac{d}{d\tau}S_\mu = \frac{e}{m}v_\mu v_\alpha F^{\alpha\nu}S_\nu \qquad . \tag{21}$$

From eq.20 it is by no means evident that the zero component of the polarization will vanish in the rest frame.

Considering that BMT refers to averages, and assuming that $\langle v_\mu v_\alpha S_\nu \rangle = \langle v_\mu v_\alpha \rangle \langle S_\nu \rangle$, eq.21 gives BMT without anomalous contribution to the magnetic moment if

$$\langle v_\mu v_\nu \rangle = g_{\mu\nu} \qquad . \tag{22}$$

This average does not seem sensible for spatial indices but this invariant expression is consistent with $\gamma^1\gamma^1 = -\alpha_x\alpha_x$ which must be used in quantum Dirac theory. This shows that BMT corresponds to averages of quantum quantities and not of classical ones.

Therefore to take radiative effects into account in a semiclassical way, our approach seems more appropriate. Instead of considering the detailed description with one velocity for charge and another for mass, we have an effective common velocity. In addition to one Bohr magneton originated from the non coincidence of charge and mass, another one comes from this common velocity which affected by the longer wavelength radiation field produces the anomalous correction.

3 Effect of large static electromagnetic fields

In our approach we must take a^μ as the average of the product of the common charge-mass instantaneous velocity and spin

$$a^\mu = \epsilon^{\mu\nu\rho\sigma}\langle u_\nu S_{\rho\sigma}\rangle \qquad . \tag{23}$$

In the global rest frame with absence of electric and magnetic fields $a^0 = \langle \mathbf{u}\cdot\mathbf{S}\rangle = 0$. The same happens when these fields are applied together with the spin produced by zitterbewegung because of the difference of scale of both motions. But if we consider the additional angular momentum caused by the precession around the magnetic field $\mathbf{B} = B\mathbf{z}$ in a range given by the Compton length

$$M_{12} = e\lambda_c^2 B \tag{24}$$

the simultaneous application of $\mathbf{E} = E\mathbf{z}$ will produce an asymmetry in the average of the velocity along this axis. This asymmetry is given by the electric force times the interval over which the average must be taken. But this interval decreases with increasing force because it must correspond to the time in which the electron may be considered globally at rest. It is reasonable to take this interval as that necessary to displace the electron by one Compton length which will be $\sim E^{-1/2}$ for non-relativistic motion and will tend to $\sim E^{-1}$ for relativistic one. Therefore we may state as a bound

$$\xi_0 \leq e\lambda_c^2 B \qquad . \tag{25}$$

It must be noticed that for a magnetic field $B \simeq 1T$, $\xi_0 \leq 10^{-10}$ which is just below the theoretical error in the radiative calculations of anomalous gyromagnetic factor.

It is interesting to see which is the correction given by a non-vanishing ξ_0 to the spin-orbit interaction as described by BMT. A lengthy but straightforward calculation for an electron with global momentum $\boldsymbol{\pi}$ under external fields $\mathbf{E}$ and $\mathbf{B}$ gives, as equation of motion for its spin

$$\begin{aligned}\frac{d\boldsymbol{\xi}}{dt} = \frac{2\mu m + 2\mu'(\epsilon - m)}{\epsilon}(\boldsymbol{\xi} \times \mathbf{B}) + \frac{2\mu'}{\epsilon(\epsilon + m)}(\boldsymbol{\pi} \cdot \mathbf{B})(\boldsymbol{\pi} \times \boldsymbol{\xi}) \\ + \frac{2\mu m + 2\mu'\epsilon}{\epsilon(\epsilon + m)}\boldsymbol{\xi} \times (\mathbf{E} \times \boldsymbol{\pi}) \\ + 2\mu'\xi_0[\mathbf{E} - \frac{(\boldsymbol{\pi} \cdot \mathbf{E})\boldsymbol{\pi}}{\epsilon(\epsilon + m)} + \frac{\boldsymbol{\pi} \times \mathbf{B}}{\epsilon}] \quad .\end{aligned} \tag{26}$$

The second line of eq.26 gives rise to the spin-orbit interaction which, in the non-relativistic limit, shows the “Thomas $\frac{1}{2}$” for normal magnetic moment and absent for its anomalous correction.

Thinking that the applied $\mathbf{B}$ in the electron rest frame may give a bound for ξ_0 as above and for $\frac{\boldsymbol{\pi}}{m} \sim 10^{-2}$, the ratio between the third line to the second one is $\leq 10^{-11}$. Since spin-orbit interaction produces a fine structure $\sim 10^{-3}eV$ the effect of ξ_0, which is of electric dipole type proportional to the anomalous part of the magnetic moment, is $\leq 10^{-14}eV$ much smaller than the experimental error in the Lamb shift $\sim 10^{-11}eV$.

Obviously it is conceivable that an application to an atom is beyond the validity range of the semiclassical approximation.

Another possible effect of ξ_0 is on the helicity change of a free electron under the influence of external uniform e.m. fields. In this case the semiclassical approximation should be valid up to high [4] magnetic fields $B < \frac{m^2}{e} \sim 10^9 T$ and even higher for very relativistic electrons. Since

$$\frac{d\xi_\parallel}{d\tau} = \frac{1}{m}(a_\parallel \frac{d\epsilon}{d\tau} - a_0 \frac{d \mid \boldsymbol{\pi} \mid}{d\tau} + \frac{da_\parallel}{d\tau}\epsilon - \frac{da_0}{d\tau} \mid \boldsymbol{\pi} \mid) \quad , \tag{27}$$

it is easy to see that the second term of eq.9 being proportional to π^μ gives no contribution to eq.27. On the other hand, with longitudinal fields, whereas the first two terms in brackets of eq.27 give $eF^{03}\xi_0$, the last two terms contribute as $-e(1 + \frac{\alpha}{\pi})F^{03}\xi_0$. Therefore

$$\frac{d\xi_\parallel}{d\tau} = \frac{e}{m}\frac{\alpha}{2\pi}E_\parallel \xi_0 \tag{28}$$

and taking the volume $V = \sqrt{1 - \frac{\pi^2}{\epsilon^2}}\lambda_c^3$ for each spin

$$\frac{1}{V}\frac{d\xi_\parallel}{dt} \leq \frac{\alpha}{2\pi}e^2 E_\parallel B_\parallel \qquad . \tag{29}$$

This may be understood as the effect of the chiral anomaly [16] on the helicity of the electron in a way analogous to the Feynman graphs analysis of ref. [13].

One must remark that $\xi_0 \neq 0$ has the consequence that from eq.9 it does not follow that $\frac{da^2}{d\tau} = 0$, and also in general $\frac{d}{d\tau} \mid \boldsymbol{\xi} \mid^2 \neq 0$. Compared to the Dirac equation, in the Foldy-Wouthuysen approach $\mid \boldsymbol{\xi} \mid^2$ is constant, but this does not happen in the 4-spinor treatment where only the 4x4 matrix $\mid \boldsymbol{\Sigma} \mid^2$ is a constant. The fact that our result gives a non constancy of $\mid \boldsymbol{\xi} \mid^2$ should be due to the inclusion in the one particle description of sea effects produced by strong external fieds.

From eq.29 for an electron traveling 1 cm at not too high velocity under longitudinal $E \approx B \sim 10^2(eV)^2 \simeq 1T$ the change of helicity would be $\leq 10^{-12}$, being necessary to take into account that for ultrarelativistic electron $\Delta\xi_\parallel \rightarrow 0$ so that the helicity cannot increase indefinitely.

We finally comment that eq.29 can also be intuitively understood as the influence of the anomalous part of the magnetic moment which allows a different precession of instantaneous velocity and spin around the magnetic field so that taking the average, the longitudinal electric field produces an asymmetry and a net variation of helicity results.

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